The space of graded traces for holomorphic VOAs of small central charge*

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February 2, 2008

Abstract

It is one of the remarkable results of vertex operator algebras (VOAs) that the graded traces (one-point correlation functions) of holomorphic VOAs are modular functions. This paper explores the question of which modular functions arise as the graded traces of holomorphic VOAs. For VOAs of small central charge, i.e., $c \leq 24$, and a non-zero weight-one subspace we find that the only conditions imposed on the modular functions are those that arise easily out of our condition that the VOAs be of CFT type, that is that they have no negative-weight subspaces and their zero-weight subspace is generated by the vacuum vector.

1 Introduction

One of the striking features of rational vertex operator algebras (VOAs) is the modularity of their graded traces, also called one-point correlation functions. Zhu proved that for any holomorphic VOA, V, satisfying a finiteness condition, and certain $v \in V$, the graded trace of v , $Z(v, \tau)$, is a meromorphic modular form, possibly with a character [Zhu96, Theorem 5.3.2]. We will refer to this result as Zhu's Theorem, although Dong, Li and Mason

*The majority of this work was part of the author's doctoral dissertation, which was completed at the University of California, Santa Cruz under the direction of Professor Geoffrey Mason.

weakened Zhu's assumptions to those stated [DLM00] [DLM96]. In this paper, we investigate an inverse to Zhu's Theorem. For a fixed VOA V, and modular form f, does there exist a $v \in V$ such that $Z(v, \tau) = f$? For the relevant definitions, please see Section 2

We consider strongly rational holomorphic VOAs of small central charge c, i.e., $c \leq 24$. We make use of work of Dong and Mason on the classification of such VOAs [DM04a]. It is well known that for holomorphic VOAs, c is positive and $8|c$. Thus we are considering the cases $c = 8$, $c = 16$ and $c = 24$. Dong and Mason prove that for $c = 8$ and $c = 16$, the only strongly holomorphic VOAs are the lattice VOAs generated by the E_8 root lattice for $c = 8$ and the lattices $E_8 + E_8$ and D_{16}^+ for $c = 16$.

However, for the $c = 24$ case Dong and Mason do not achieve a complete classification. They do show that the weight-one subspace V_1, which forms a Lie algebra, is either zero, abelian or semisimple. In this paper we treat the V_1 abelian and V_1 semisimple cases. Note that we do not use Schellekens classification of $c = 24$ holomorphic VOAs, which provides a list of 71 possible Lie algebras for V_1 [Sch93], as his classification is not mathematically rigorous.

Frenkel, Lepowsky and Meurmann conjecture that the only holomorphic $c = 24$ VOA with $V_1 = 0$ is the moonshine module [FLM88]. If this conjecture holds, then the current paper and Dong and Mason's paper *Monstrous moonshine of higher weight* find the space of graded traces for all strongly rational holomorphic VOAs of small central charge [DM00].

In this work we make the assumptions about our VOA V used in Zhu's Theorem, i.e., holomorphic and C_2-cofinite, which are explained in Section 2. Additionally, we assume that the VOAs are of CFT-type (here CFT stands for conformal field theory). This assumption places immediate restrictions on the set of modular forms achieved as graded traces of the elements of V. The main result of this paper is that for small central charge these are the only restrictions.

Let M_k denote the space of holomorphic modular forms of weight k and let $\eta(\tau)$ be the Dedekind eta function; see equation (9).

Theorem 1.1. *Let V be a strongly rational holomorphic vertex operator algebra with central charge c equal to* 8 *or* 16*. For any holomorphic modular form $f(\tau)$ with* $\mathrm{wt}\, f \geq c/2$*, there exists an $v \in V$ such that*

$$Z(v, \tau) = \frac{f(\tau)}{\eta(\tau)^c}.$$

This means that every possible modular form is attained as the graded trace of some element of v. Combining this theorem with Zhu's Theorem characterizes the space of graded traces.

Corollary 1.2. *Let V be a strongly rational holomorphic vertex operator algebra with central charge c equal to 8 or 16. The space of graded traces of V is*

$$\frac{1}{\eta(\tau)^c}\bigoplus_{k\geq c/2} M_k.$$

For $c = 24$ we attain the following theorem.

Theorem 1.3. *Let V be a strongly rational holomorphic vertex operator algebra with central charge $c = 24$ and $V_1 \neq 0$. Then for any holomorphic modular form $f(\tau)$ with $\mathrm{wt}\, f \geq 14$, there exists a vector v in V such that*

$$Z(v,\tau) = \frac{f(\tau)}{\Delta(\tau)}.$$

Here $\Delta(\tau) = \eta(\tau)^{24}$ is the discriminant function. With Zhu's Theorem, this describes the space of graded traces for v with square-bracket weight greater than zero. It remains to consider the vacuum vector, $\mathbf{1}$. In the $c = 24$ case, $Z(\mathbf{1},\tau)$ is a modular function with a pole of order 1 at infinity and no poles in the upper half plane. The modular functions are rational functions in the modular invariant $J(\tau)$. If normalize the q-expansion of $J(\tau)$ to be $q^{-1} + 0 + 196884q + \cdots$, then $Z(\mathbf{1},\tau) = \dim V_1 + J(\tau)$, and we can describe the space of graded traces of V.

Corollary 1.4. *Let V be a strongly rational holomorphic vertex operator algebra with central charge 24 and $V_1 \neq 0$. The space of graded traces of V is*

$$\mathbb{C}(\dim V_1 + J(\tau)) \oplus \frac{1}{\Delta(\tau)}\bigoplus_{k\geq 14} M_k.$$

2 Definitions

Let $(V, Y, \omega, \mathbf{1})$ be a vertex operator algebra. We recall the portions of the definition of a VOA relevant to this paper. For the full axioms, see Frenkel, Huang and Lepowsky [FHL93]. The vector space V is $\mathbb{Z}$-graded:

$V = \bigoplus_{n \geq n_0} V_n$ and elements of V_n are called homogeneous of weight n. We denote the vertex operator $Y(v,z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}$. Here each $v(n)$ is an endomorphism of V. The most important VOA identity is the Jacobi identity, which we do not state here; however, in this paper, we will make frequent use of associativity,

$$(u(m)v)(n) = \sum_{i \geq 0} (-1)^i \binom{m}{i} \big(u(m-i)v(n+i) - (-1)^m v(m+n-i)u(i) \big), \quad (1)$$

which follows directly from the Jacobi identity.

There is a distinguished element $\omega \in V_2$, called the conformal vector. Its modes are denoted by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$. Clearly $L(n) = \omega(n+1)$. The redundant notation is introduced as the $L(n)$ operators make V a module for the Virasoro Lie algebra. That is

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} \operatorname{id} c.$$

The constant c is called the *central charge* of V. The $L(0)$ operator determines the grading of V, i.e., for $v \in V_n$, $L(0)v = nv$. The highest-weight vectors for this representation of the Virasoro algebra, i.e., those $v \in V$ such that $L(n)v = 0$ for all $n > 0$, are called the *highest-weight vectors* of V.

For homogeneous $v \in V_n$, the *zero mode* of v is defined $o(v) := v(n-1)$. By extending linearly, we can define $o(v)$ for all $v \in V$. The zero mode preserves the grading of V. The graded trace of v on the VOA is defined by:

$$Z(v,q) = q^{-c/24} \sum_{n \in \mathbb{Z}} \operatorname{tr}\big|_{V_n} o(v) q^n.$$

Here q is just a formal variable; however, later we will consider q as a complex variable of modulus less than one. As usual in the theory of modular forms, we define τ in the complex upper half plane by $q = e^{2\pi i \tau}$. Hence we also use $Z(v, \tau)$ to denote the graded trace.

Let V be a vertex operator algebra. We say that V is *rational*, if every admissible module for V is completely reducible. [DLM98] It is *holomorphic*, if it is rational and has only one irreducible module. Furthermore, we say that V is C_2*-cofinite*, if the subspace generated by $\{u(-2)v | u, v \in V\}$ is of finite codimension in V. It is conjectured that rational implies C_2-cofinite.

A VOA is of *CFT-type*, if there are no negatively graded subspaces and the zero-graded space is one dimensional and generated by the vacuum vector,

i.e.,

$$V = \mathbb{C}\mathbf{1} + V_1 + V_2 + \cdots .$$

To be of *strong CFT-type*, we additionally require that $L(1)V_1 = 0$. For holomorphic VOAs, CFT-type implies strong CFT-type [DM04b]. Following the terminology of Dong and Mason, we call rational C_2-cofinite VOAs of strong CFT-type *strongly rational*. This paper concerns strongly rational holomorphic VOAs with central charge $c \leq 24$.

To study the graded traces of a VOA $(V, Y(\ , z), \omega, \mathbf{1})$, Zhu introduced another VOA $(V, Y[\ , z], \tilde{\omega}, \mathbf{1})$, which is isomorphic to $(V, Y(\ , z), \omega, \mathbf{1})$ [Zhu96]. While the underlying vector space and the vacuum vector are the same as the original VOA, the vertex operator is defined $Y[v, z] := Y(v, e^z - 1)e^{z \operatorname{wt} v}$, and the conformal vector is $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. This induces a different grading $V = \sum_{n \geq n_0} V_{[n]}$. We denote the square-bracket vertex operators by $Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1}$. Let $v \in V_m$. To expand the square-bracket operator $v[n]$ in terms of round-bracket operators and vice-versa we use:

$$v[n] = \operatorname{Res}_z Y(v, z)(\log(1 + z))^n (1 + z)^{m-1}, \tag{2}$$

$$v(n) = \operatorname{Res}_z Y[v, z](e^z - 1)^n e^{z(1-m)} \tag{3}$$

which follow from the definition of $Y[v, z]$. Using these formulas, we can show that $\bigoplus_{n \leq N} V_{[n]} = \bigoplus_{n \leq N} V_n$.

3 Modular Forms and Zhu's Theorem

Throughout this section V^c will denote a holomorphic, C_2-cofinite vertex operator algebra of CFT-type with central charge c. For $c = 24$, we also assume that the weight-one homogeneous space is non-zero, i.e., $V_1^{24} \neq 0$.

Let $v \in V_{[k]}^c$. From Zhu we know that $Z(v, q)$ the graded trace of v converges to $q^h f(q)$ for some f a holomorphic function on $D = \{q \in \mathbb{C} \mid |q| < 1\}$ [Zhu96, Theorem 4.4.1]. Additionally, Zhu shows that the graded trace of v is a modular form of weight k in the following sense: Change variable to τ, via $q = e^{2\pi i \tau}$. then for all $v \in V_{[k]}$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$

$$Z(v, g \cdot \tau) = \psi(g)(c\tau + d)^k Z(v, \tau), \tag{4}$$

where $g \cdot \tau$ is the usual action of g by linear fraction transformations, and $\psi : PSL_2(\mathbb{Z}) \to \mathbb{C}^*$ is a group character [Zhu96, Theorem 5.3.3].

For this paper, in addition to V^c holomorphic and C_2-cofinite we assume that V^c is of CFT-type. So

$$Z(v,q) = q^{-c/24} \sum_{n \geq 0} \operatorname{tr}\big|_{V_n} o(v) q^n. \tag{5}$$

Thus there exists an f, holomorphic on D, such that

$$Z(v,q) = q^{-c/24} f(q).$$

Furthermore, because the abelianization of $PSL_2(\mathbb{Z})$ is a cyclic group of order six generated by the coset of $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, equation (5) allows us to determine the character in equation (4). Let χ be the character of $PSL_2(\mathbb{Z})$ determined by $\chi(T) = e^{2\pi i/6}$, then

$$Z(v, g \cdot \tau) = \chi(g)^{-c/4} (c\tau + d)^k Z(v, \tau),$$

for all $v \in V_{[k]}$.

Let $P_k^{-c/24}$ denote the space of functions h on D such that there exists a function f holomorphic on D with

$$h(q) = q^{-c/24} f(q) \tag{6}$$
$$h(g \cdot \tau) = \chi(g)^{-c/4} (c\tau + d)^k h(\tau), \tag{7}$$

for all $g \in PSL_2(\mathbb{Z})$ and χ as above. We refer to the elements of $P_k^{-c/24}$ as weight-k modular forms with a pole of order $c/24$ at infinity. Let $Z(W) := \{Z(v,q) | v \in W\}$ be the space of graded traces for W. Using this notation, Zhu's Theorem implies that $Z(V^c_{[k]}) \subseteq P_k^{-c/24}$ for all integers k.

Note that the usual definition of a modular form f of weight k requires that f be holomorphic on D and have trivial character. We will call such forms *holomorphic modular forms* of weight k and denote their space by M_k. We recall some basic results about holomorphic modular forms. For proofs and further discussion see Serre's *A Course in Arithmetic* or another introductory text on modular forms [Ser73].

There are no non-zero holomorphic modular forms with odd or negative integral weight, and the only weight-zero forms are the constant functions. There are also no non-zero holomorphic modular forms of weight two, but for every even integer $k \geq 4$, there exists a form, G_k, called the Eisenstein

series of weight-k. The Eisenstein series can be defined by their expansions at infinity.

$$G_k(q) := -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \tag{8}$$

Here B_k is the kth Bernoulli number defined by $\frac{t}{e^t-1} = \sum_{k\geq 0} B_k \frac{t^k}{k!}$, and $\sigma_\ell(n) := \sum_{d|n} d^\ell$.

A key subspace of M_k is the space of cusp forms

$$M_k^0 := \Big\{ \sum_{n\geq 0} a_n q^n \in M_k \Big| a_0 = 0 \Big\}.$$

The smallest integer k such that $\dim M_k^0 > 0$ is 12, and M_{12}^0 is spanned by the discriminant, $\Delta := 10,800\,(20(G_4)^3 - 49(G_6)^2)$, which has a simple zero at $\tau = \infty$ ($q = 0$) and no other zeros or poles. The discriminant has particularly nice product formula:

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

and multiplication by the discriminant is a linear isomorphism of M_k and M_{k+12}^0.

In general multiplication of two forms of weights k and ℓ gives a form of weight $k+\ell$. This makes the space of all holomorphic modular forms $M := \oplus_{k\geq 0} M_k$ a graded algebra. Indeed, it is a polynomial algebra generated by G_4 and G_6.

The 24th root of the discriminant is the Dedekind eta function

$$\eta(q) := q^{1/24}\prod_{n>0}(1-q^n). \tag{9}$$

(See for example Lang's book on modular forms [Lan76].) The elements $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ generate the modular group, $PSL_2(\mathbb{Z})$. The eta function is a modular form of weight $1/2$ and multiplier system ϕ defined by $\phi(T) = e^{2\pi i/24}$ and $\phi(S) = \sqrt{\frac{1}{i}}$, i.e., for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$,

$$\eta(g\cdot\tau) = \phi(g)(c\tau + d)^{1/2}\eta(\tau). \tag{10}$$

In the same way that multiplication by the discriminant gives a linear isomorphism of M_k and M_{k+12}^0, multiplication by $\eta(\tau)^c$ gives a linear isomorphism of $P_k^{-c/24}$ and $M_{k+c/2}$.

Proposition 3.1. *Let $8|c$, multiplication by $\eta(\tau)^c$ is a linear isomorphism of $P_k^{-c/24}$ and $M_{k+c/2}$.*

Proof. Let $h(\tau)$ be in $P_k^{-c/24}$. Since $h(\tau)$ and $\eta(\tau)$ are holomorphic for all finite τ, the product $\eta(\tau)^c f(\tau)$ is holomorphic except possibly at infinity. From the q-expansions, equations (6) and (9), the product is also holomorphic at infinity. From equations (7) and (10), for any $g \in PSL_2(\mathbb{Z})$,

$$\eta(g \cdot \tau)^c h(g \cdot \tau) = \phi(g)^c \chi(g)^{-c/4} (c\tau + d)^{k+c/2} \eta(\tau)^c h(\tau).$$

We compute that $\phi(T)^c \chi(T)^{-c/4} = 1$. Since $8|c$, $\phi(S)^c = 1$, and since $\chi(S) = 1$, $\chi(S)^{-c/4} = 1$. So $\phi(g)^c \chi(g)^{-c/4} = 1$ for all $g \in PSL_2(\mathbb{Z})$ Thus $\eta(\tau)^c h(\tau)$, is in $M_{k+c/2}$.

It is clear that multiplication by $\eta(\tau)^c$ is linear. Moreover, because $\eta(\tau)$ has q-expansion $\eta(q) = q^{1/24} f(q)$, such that $f(q)$ has no zero or poles in D, division by $\eta(\tau)^c$ maps $M_{k+c/2}$ back to $P_k^{-c/24}$. Thus multiplication by $\eta(\tau)^c$ is invertible, and an isomorphism. $\square$

Using this proposition, Zhu's Theorem implies that

$$Z(V^c_{[k]}) \subseteq \frac{1}{\eta(\tau)^c} M_{k+c/2}.$$

Since assuming that V^c is holomorphic implies the central charge c is an integer divisible by 8, $k + c/2$ is an integer. Because V^c is of CFT-type, $Z(V^c) \subseteq \bigoplus_{k \geq 0} P_k^{-c/24}$. Thus

$$Z(V^c) \subseteq \frac{1}{\eta(\tau)^c} \bigoplus_{k \geq c/2} M_k. \tag{11}$$

For $c = 8$ and $c = 16$ our main theorem, Theorem 1.1, says that all such modular forms are achieved as the graded traces of elements of V^c. So our theorem and Zhu's Theorem imply Corollary 1.2, i.e., that

$$Z(V^c) = \frac{1}{\eta(\tau)^c} \bigoplus_{k \geq c/2} M_k.$$

For $c = 24$ and $k = 0$, Zhu's Theorem says that $Z(V^{24}_{[0]}) \subseteq \frac{1}{\Delta(\tau)} M_{12}$. In this case, CFT-type imposes another another restriction on the image of $V^{24}_{[0]}$.

Since $\dim V_{[0]}^{24} = 1$ and $\dim M_{12} = 2$, $Z(V_{[0]}^{24})$ is a proper subspace of $\frac{1}{\Delta(\tau)} M_{12}$. It is generated by the graded trace of the vacuum vector, $\mathbf{1}$. Thus

$$Z(V^{24}) \subseteq \mathbb{C}Z(\mathbf{1}, \tau) \oplus \frac{1}{\Delta(\tau)} \bigoplus_{k \geq 14} M_k. \tag{12}$$

Our main theorem for $c = 24$, Theorem 1.3, says that $\frac{1}{\Delta(\tau)} \bigoplus_{k \geq 14} M_k \subseteq Z(V^{24})$. So equation (12) is actually an equality. This is essentially Corollary 1.4. For the corollary, note that from Proposition 3.1, $Z(\mathbf{1}, \tau) \in P_{[0]}^{-1} = \frac{1}{\Delta(\tau)} M_{12}$, which has basis the constant function 1 and the modular invariant $J(q) = q^{-1} + 196884q + \cdots$. Moreover $Z(\mathbf{1}, q) = q^{-1} + \dim V_1 + \dim V_2 q + \cdots$. So $Z(\mathbf{1}, \tau) = \dim V_1 + J(\tau)$, and we get Corollary 1.4 exactly.

4 Proof of Main Theorems

To prove our main theorems, we will use several results that Dong and Mason developed to determine the space of graded traces of the moonshine module. They state their results for the moonshine module, but the same proofs work for the more general statements we give here.

Proposition 4.1 ([DM00]). *Let V be a strongly rational holomorphic VOA. For any $\ell \geq 0$, the graded trace of $L[-2]^\ell \mathbf{1} \in V_{[2\ell]}$ is non-zero. More specifically,*

$$Z(L[-2]^\ell \mathbf{1}, \tau) = q^{-c/24} \sum_{n \geq 0} a_n q^n,$$

such that $a_0 \neq 0$.

The above proposition implies that $\eta(\tau)^c Z(L[-2]^\ell \mathbf{1}, \tau)$ is in $M_{2\ell + c/2}$ but not in $M^0_{2\ell + c/2}$. Since $\dim M_k / M_k^0 \leq 1$, this means that for all integers $k \geq c/2$,

$$M_k = \mathbb{C}\eta(\tau)^c Z(L[-2]^{(2k-c)/4} \mathbf{1}, \tau) \oplus M_k^0. \tag{13}$$

We have restricted our problem to determining for which cusp forms $f(\tau) \in M_k^0$, $k \geq c/2$, there exists a $v \in V^c$ such that $\eta(\tau)^c Z(v, \tau) = f(\tau)$.

For $c = 8$, this question has already been answered. The unique strongly rational holomorphic VOA with central charge 8 is the E_8 root lattice VOA, V_{E_8} [DM04a]. For this VOA, Dong Mason and Nagatomo use a result of

Waldspurger to show that for all cusp forms $f(\tau)$, there exists a highest-weight vector v such that $\eta(\tau)^8 Z(v,\tau) = f(\tau)$ [DMN01]. Thus $m_8 \circ Z$ maps V_{E_8} onto $\bigoplus_{k\geq 4} M_k$. This proves Theorem 1.1 for $c = 8$.

For the remaining central charges we use the following key lemma.

Lemma 4.2 (main lemma). *Let V be a strongly rational holomorphic VOA with central charge 16 or 24 and $V_1 \neq 0$. Then there exists a $v \in V_{[4]}$ such that*

$$\operatorname{tr} o(v)\big|_{V_0} = 0,$$
$$\operatorname{tr} o(v)\big|_{V_1} \neq 0.$$

The proof of Lemma 4.2 is quite lengthy and it is postponed to Section 6. The lemma is used to prove the following proposition.

Proposition 4.3. *Let V^c be a strongly rational holomorphic VOA with central charge c equal to 16 or 24 and $(V^c)_1 \neq 0$. There exists a $v \in (V^{16})_{[4]}$ such that*

$$Z(v,\tau) = \eta(\tau)^8,$$

and there exists a $v \in (V^{24})_{[4]}$ such that

$$Z(v,\tau) = G_4(\tau).$$

Proof. Assuming Lemma 4.2 gives the existence of a $v \in V^c$ such that $\eta(\tau)^c Z(v,\tau)$ is a non-zero element of $M^0_{4+c/2}$. For $c = 16$, this is M^0_{12}, which is a one-dimensional space generated by $\Delta(\tau) = \eta(\tau)^{24}$. Thus, adjusting by a constant multiple if necessary, $\eta(\tau)^{16} Z(v,\tau) = \Delta(\tau)$, i.e.

$$Z(v,\tau) = \frac{\Delta(\tau)}{\eta(\tau)^{16}} = \eta(\tau)^8.$$

For $c = 24$, we have that $\Delta(\tau) Z(v,\tau)$ is a non-zero element of M^0_{16}, which is also one dimensional and generated by $\Delta(\tau) G_4(\tau)$. So adjusting by a constant if necessary,

$$Z(v,\tau) = \frac{\Delta(\tau) G_4(\tau)}{\eta(\tau)^{24}} = G_4(\tau).$$

$\square$

We do not need prove a lemma like Lemma 4.2 for $v \in V_{[k]}$, $k > 4$. For those vectors we use a result of Dong and Mason which describes the space of graded traces for the Virasoro module generated by a highest-weight vector w.

To describe this space of graded traces, we use a derivation $\delta_k : P_k^{-c/24} \to P_{k+2}^{-c/24}$. This derivation is a generalization of the derivation $\delta_k : M_k \to M_{k+2}$ described in Lang. [Lan76][Chapter X, Section 5]

Define the "Eisenstein series" $G_2(\tau)$ by

$$G_2(q) = -\frac{1}{12} + 2\sum_{n=1}^{\infty} \sigma_1(n)\, q^n.$$

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$,

$$G_2(g \cdot \tau) = (c\tau + d)^2 G_2(\tau) - \frac{c}{2\pi i}(c\tau + d).$$

Thus G_2 is not a modular form. For $f(q) \in P_k^{-c/24}$, define

$$\delta_k f(q) = q\frac{d}{dq} f(q) + kG_2(q)f(q).$$

Proposition 4.4. *The map δ_k takes $P_k^{-c/24}$ to $P_{k+2}^{-c/24}$ and is a derivation in the sense that if $g \in M_k$ and $f \in P_\ell^{-c/24}$, then*

$$\delta_{k+\ell}(gf) = \big(\delta_k(g)\big)f + g\big(\delta_\ell(f)\big).$$

Proof. The proof relies on $\delta_k : M_k \to M_{k+2}$ being a derivation of the space of holomorphic modular forms, and equation (14), below.

The logarithmic derivative of $\eta(\tau)$ is $-\pi i G_2(\tau)$ and $q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Using these facts, the definition of δ_k and the quotient rule, we show that if $f(\tau) \in P_k^{-c/24}$ equals $\frac{h(\tau)}{\eta(\tau)^c}$ with $h(\tau) \in M_{k+c/2}$, then

$$\delta_k(f(\tau)) = \frac{\delta_{k+c/2}(h(\tau))}{\eta(\tau)^c}. \tag{14}$$

Since $\delta_{k+c/2}(h(\tau)) \in M_{k+c/2+2}$, $\delta_k(f(\tau)) \in P_{k+2}^{-c/24}$. To see that δ has the derivation property given in the proposition, simply use equation (14) and that δ is a graded derivation on the space of holomorphic modular forms. $\square$

Define a δ-submodule of $P^{-c/24}$ to be a linear subspace closed under multiplication by elements of M and the action of δ_k.

Theorem 4.5 ([DM00]). *Let V be a C_2-cofinite holomorphic VOA and let w be a highest-weight vector of positive weight. The space of graded traces consisting of all $Z(v,\tau)$ for v in the Virasoro module generated by w is the δ-module generated by $Z(w,\tau)$.*

Thus if there exists a highest-weight vector $w \in V^{24}$ with graded trace equal to $G_4(\tau)$, then $Z(V^{24})$ contains the δ-module generated by $G_4(\tau)$. The δ-module generated by $G_4(\tau)$ also contains $G_6(\tau)$, as $\delta_4 G_4 = 14G_6$ [Lan76]. Since G_4 and G_6 generate the space of holomorphic forms as a polynomial algebra, the δ-module generated by $G_4(\tau)$ contains all the holomorphic modular forms except the constant functions, which implies that

$$\bigoplus_{k>0} M_k \subseteq Z(V^{24}).$$

In light of equation (13), this would complete the proof of the main theorem for the $c = 24$ case, Theorem 1.3.

If there exists a highest-weight vector $w \in V^{16}$ with graded trace equal to $\eta(\tau)^8$, then $Z(V^{16})$ contains the δ-module in $P^{-2/3}$ generated by $\eta(\tau)^8$. Thus $m_{16} \circ Z(V^{16})$ contains the image of that δ-module under m_{16}, which contains $\eta(\tau)^{16}\eta(\tau)^8 = \Delta(\tau)$. The image of a δ-module in $P^{-2/3}$ under m_{16} is a δ-module in M. The M-module generated by Δ in M is M^0, the space of cusp forms. So $M^0 \subseteq m_{16} \circ Z(V^{16})$. Combining this with equation (13) proves Theorem 1.1 in the $c = 16$ case.

Note that while Proposition 4.3 gives the existence of vectors $v_1 \in V^{24}$ and $v_2 \in V^{16}$ with $Z(v_1,\tau) = G_4(\tau)$ and $Z(v_2,\tau) = \eta(\tau)^8$, the above argument requires the existence of *highest-weight* vectors with those graded traces. To fill this hole in our argument we need the following lemma.

Lemma 4.6. *Let V^c be a strongly rational holomorphic VOA with central charge $c = 16$ or 24. If there exists a $v \in V_{[4]}$ such that $\operatorname{tr} o(v)|_{V_0} = 0$, then there exists a highest-weight vector w in V^c such that $Z(w,\tau) = Z(v,\tau)$.*

We prove this lemma in the following section and the main lemma, Lemma 4.2, in Section 6. This will complete the proof of our main theorem.

5 Proof of Lemma 4.6

By a highest-weight vector in Lemma 4.6, we mean a round-bracket highest-weight vector, i.e., $L(n)v = 0$ for all $n > 0$. However, using equations (2) and (3) it is not hard to show that the set of round-bracket highest-weight vectors in V_4^c equals the set of square-bracket highest-weight vectors in $V_{[4]}^c$.

To analyze $V_{[4]}^c$ we make use of the symmetric invariant bilinear form on square-bracket V^c. A bilinear form $(\ ,\ )$ on V is said to be *invariant* if

$$\big(Y[u,z]v, w\big) = \big(v, Y(e^{zL[1]}(-z^{-2})^{L[0]}u, z^{-1})w\big), \tag{15}$$

for all $u, v, w \in V^c$ [FHL93]. Li proved that the space of invariant forms on V is isomorphic to $\big(\frac{(V^c)_{[0]}}{L[1](V^c)_{[1]}}\big)^*$ [Li94]. Since V^c is of strong CFT-type, $\big(\frac{(V^c)_{[0]}}{L[1](V^c)_{[1]}}\big)^*$ is one-dimensional, and the invariant bilinear form on V^c is unique up to a scalar. Henceforth, we fix $(\ ,\ )$ to be the unique invariant form on V^c so that $(\mathbf{1}, \mathbf{1}) = 1$. Because V^c is holomorphic, it is simple, and $(\ ,\ )$ is non-degenerate.

Harada and Lam showed that $V_{[4]}^c = \operatorname{Ker} L[1]\big|_{V_{[4]}^c} \oplus L[-1]V_{[3]}^c$ and $V_{[4]}^c = \operatorname{Ker} L[2]\big|_{V_{[4]}^c} \oplus L[-2]V_{[2]}^c$, both sums orthogonal with respect to the square-bracket invariant bilinear form on V^c [HL95]. Because the form is non-degenerate, it follows that

$$V_{[4]}^c = (\operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]) \oplus (L[-1]V_{[3]}^c + L[-2]V_{[2]}^c). \tag{16}$$

We will show that for any $v \in V_{[4]}^c$ such that $\operatorname{tr}|_{V_0^c} o(v) = 0$, then there exists an element of $\operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]$ with the same graded trace. Because $\operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]$ is precisely the space of square-bracket highest-weight vectors in $V_{[4]}^c$, this will prove Lemma 4.6.

From equation (16), there exist $w \in \operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]$ and $u \in L[-1]V_{[3]}^c + L[-2]V_{[2]}^c$ such that $v = w + u$. Assume that $\operatorname{tr}|_{V_0^c} o(v) = 0$. Since w is a highest-weight vector, $w \in V_4^c$, and from the creation axiom: $\operatorname{tr}|_{V_0^c} o(w) = 0$. Therefore

$$\operatorname{tr}\big|_{V_0^c} o(u) = 0.$$

Now let $u = L[-1]a + L[-2]b$ with $a \in V_{[3]}^c$ and $b \in V_{[2]}^c$. Because $Z(L[-1]a, \tau) = 0$ for all $a \in V^c$ ([Zhu96] and [DLM00]), $\operatorname{tr}|_{V_0^c} o(L[-1]a) = 0$. Thus

$$\operatorname{tr}\big|_{V_0^c} o(L[-2]b) = 0.$$

Furthermore $Z(L[-1]a,\tau)=0$ implies that

$$Z(v,\tau)=Z(w,\tau)+Z(L[-2]b,\tau),$$

Hence the following claim completes the proof of Lemma 4.6.

Claim 5.1. *Suppose that V^c is a strongly rational holomorphic VOA with central charge $c=16$ or 24. Let $b\in V^c_{[2]}$. If $\operatorname{tr}|_{V_0^c}o(L[-2]b)=0$, then $Z(L[-2]b,\tau)=0$.*

Proof. Write b in terms of round brackets, $b=b_2+b_1+b_0$, $b_i\in V_i$. In the square bracket VOA, the conformal element is $\tilde{\omega}=\omega-\frac{c}{24}\mathbf{1}$. Using equation (2), we compute

$$L[-2]=-\frac{c}{24}\operatorname{id}+L(-2)+\frac{3}{2}L(-1)+\frac{5}{12}L(0)-\frac{1}{24}L(1)+\frac{11}{720}L(2)+\cdots.$$

From the creation axiom,

$$\operatorname{tr}\big|_{V_0^c}o(L[-2]b)=\operatorname{tr}\big|_{V_0^c}(L[-2]b)_0(-1),$$

where $(L[-2]b)_0$ is the projection of $L[-2]b$ into V_0^c. Since $L(0)b_0=0$ and $L(1)b_1=0$ as V is strong CFT-type, $(L[-2]b)_0=-\frac{c}{24}b_0+\frac{11}{720}L(2)b_2$. Thus

$$\operatorname{tr}\big|_{V_0^c}o\Big(-\frac{c}{24}b_0+\frac{11}{720}L(2)b_2\Big)=0,$$

and because V is of CFT-type, $-\frac{c}{24}b_0+\frac{11}{720}L(2)b_2=0$. Furthermore $L[2]=L(2)-(1/2)L(3)+\cdots$, so $L[2]b=L(2)b_2$. Thus

$$L[2]b=\frac{30c}{11}b_0. \tag{17}$$

We now turn to the following equation from Zhu [Zhu96], which holds for C_2-cofinite VOAs – see also [DLM00, eq. 5.8]. For $b\in V_{[2]}$,

$$Z(L[-2]b,\tau)=\delta Z(b,\tau)+G_4(\tau)Z(L[2]b,\tau). \tag{18}$$

Here δ denotes δ_k extended linearly to all $P_k^{-c/24}$. Note that if the coefficient of $q^{-c/24}$ in $Z(b,\tau)$ is zero, then the coefficient of $q^{-c/24}$ in $\delta Z(b,\tau)$ is also

zero. Let ℓ be defined so that $b_0 = \ell \mathbf{1}$, then, from equations (17) and (18),

$$\begin{aligned} Z(L[-2]b, \tau) &= \delta Z(b_2 + b_1, \tau) + \delta Z(b_0, \tau) + \frac{30c}{11} G_4(\tau) Z(b_0, \tau), \\ &= 0 q^{-c/24} + \cdots + -\frac{\ell c}{24} q^{-c/24} + \cdots \\ &\quad + \frac{30c}{11}\Big(\frac{1}{720} + \frac{1}{3}\sum_{n>0} \sigma_3(n) q^n\Big)(\ell q^{-c/24} + \cdots). \end{aligned}$$

Equating the coefficients of $q^{-c/24}$ gives

$$\mathrm{tr}\,\big|_{V_0} o(L[-2]b) = -\frac{10\ell c}{11}.$$

Hence $\frac{10\ell c}{11} = 0$ and since $c \neq 0$, $\ell = 0$, that is $b_0 = 0$, and from equation (17), $L[2]b = 0$. Additionally, $b_0 = 0$ implies that $\mathrm{tr}\,|_{V_0} o(b) = 0$. Hence $\eta(\tau)^c Z(b, \tau)$ is a cusp form of weight $4 + c/2$. Since $c = 16$ or 24, $\eta(\tau)^c Z(b, \tau)$ is in M_{10}^0 or M_{14}^0. There are no non-zero cusp forms of weight 10 or 14, so $Z(b, \tau) = 0$. Using this and $L[2]b = 0$, in equation (18), yields $Z(L[-2]b, \tau) = 0$ as claimed. □

Remark 5.2. *The same result and proof holds for $c = 8$ as well. However, in that case, $Z(v, \tau) = 0$ for all $v \in V^8_{[4]}$ such that $\mathrm{tr}\,|_{V^8_0} o(v) = 0$, so the result is inconsequential.*

6 Proof of Lemma 4.2

Let V be a VOA of CFT type, then V_1 is a Lie algebra with an invariant symmetric bilinear form. For a and b elements of V_1, the Lie bracket is given by $[a, b] = a(0)b$, and the bilinear form is given by $\langle a|b\rangle \mathbf{1} = a(1)b$, where $\mathbf{1}$ is the vacuum vector. We refer to $\langle\ |\ \rangle$ as the Li-Zamilodichov metric. The proof that V_1 is a Lie algebra depends on skew-symmetry (see [FHL93]) and associativity, equation (1).

The operators $a(m), b(n) \in \mathrm{End}(V)$ make V a module for an affine Lie algebra, i.e, they satisfy the commutation relations

$$[a(m), b(n)] = [a, b](m + n) + m\langle a|b\rangle \,\mathrm{id}\, \delta_{m+n,0}. \tag{19}$$

This is proved using the Jacobi identity.

To prove the main lemma, Lemma 4.2, we will explicitly compute the traces of certain elements of $V_{[4]}$ on V_0 and V_1. For V of strong CFT-type, $V_1 = V_{[1]}$, and, if a, b, c and d are in V_1, then $a[-1]b[-1]c[-1]d$ is in $V_{[4]}$.

Lemma 6.1. *Let V be a strongly rational vertex operator algebra. Let a, b, c and d be elements of V_1 and n be the dimension of V_1.*

$$\operatorname{tr}\Big|_{V_0} o(a[-1]b[-1]c[-1]d) = \tag{20}$$
$$-\frac{1}{720}\big(4\langle[a,b]|[c,d]\rangle + 5\langle[a,d]|[b,c]\rangle\big) + \frac{1}{1152}\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\langle\pi(c)|\pi(d)\rangle$$

$$\operatorname{tr}\Big|_{V_1} o(a[-1]b[-1]c[-1]d) = \tag{21}$$
$$-\frac{n+240}{720}\big(4\langle[a,b]|[c,d]\rangle + 5\langle[a,d]|[b,c]\rangle\big) + \frac{n-48}{1152}\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\langle\pi(c)|\pi(d)\rangle$$
$$-\frac{1}{48}\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\kappa(\pi(c),\pi(d)) + \frac{1}{6}\operatorname{tr}\Big|_{V_1}\sum_{\pi\in S_3} a(0)\pi\big(b(0)\big)\pi\big(c(0)\big)\pi\big(d(0)\big)$$

Proof. Because the graded trace is defined in terms of round-bracket homogeneous vectors, we need to expand $a[-1]b[-1]c[-1]d$ in terms of round brackets. We apply equation (2),

$$v[m] = \operatorname{Res}_z Y(v,z)(\log(1+z))^m(1+z)^{\operatorname{wt} v-1},$$

repeatedly to get

$$\begin{aligned}
a[-1]b[-1]c[-1]d = {} & a(-1)b(-1)c(-1)d \\
& +\tfrac{1}{2}\Big(a(0)b(-1)c(-1)d + a(-1)b(0)c(-1)d + a(-1)b(-1)[c,d]\Big) \\
& -\tfrac{1}{12}\Big(a(1)b(-1)c(-1)d + a(-1)b(1)c(-1)d + \langle c|d\rangle a(-1)b\Big) \\
& +\tfrac{1}{4}\Big(a(0)b(0)c(-1)d + a(0)b(-1)[c,d] + a(-1)[b,[c,d]]\Big) \\
& +\tfrac{1}{24}\Big(a(2)b(-1)c(-1)d + a(-1)b(2)c(-1)d - a(1)b(0)c(-1)d \\
& -a(0)b(1)c(-1)d - a(1)b(-1)[c,d] - \langle b|[c,d]\rangle a - \langle c|d\rangle[a,b] \\
& +3[a,[b,[c,d]]]\Big) - \tfrac{19}{720}a(3)b(-1)c(-1)d \\
& +\tfrac{1}{48}\Big(a(2)b(0)c(-1)d + a(2)b(-1)[c,d] - \langle[a,b]|[c,d]\rangle\mathbf{1}\Big) \\
& +\tfrac{1}{144}\Big(a(1)b(1)c(-1)d + \langle c|d\rangle\langle a|b\rangle\mathbf{1}\Big).
\end{aligned} \tag{22}$$

Thus the square-bracket weight-four vector, $a[-1]b[-1]c[-1]d$ is the sum of round-bracket vectors of weights zero through four. To compute its traces on V_0 and V_1, we compute the traces of each round-bracket homogeneous component separately and add the result. Properly, we are actually computing the traces of the zero mode of each vector; however, we usually neglect to mention "zero mode" explicitly. We start with the round-bracket weight-zero terms, namely:

$$-\frac{19}{720}a(3)b(-1)c(-1)d+\frac{1}{48}\big(a(2)b(0)c(-1)d+a(2)b(-1)[c,d]-\langle[a,b]|[c,d]\rangle\mathbf{1}\big) + \frac{1}{144}\big(a(1)b(1)c(-1)d + \langle c|d\rangle\langle a|b\rangle\mathbf{1}\big). \quad (23)$$

Using the affine Lie algebra commutation relations and the invariance of the bilinear form, we can show that the weight-zero terms, (23), equal

$$\big(-\frac{1}{180}\langle[a,b]|[c,d]\rangle-\frac{1}{144}\langle[a,d]|[b,c]\rangle+\frac{1}{144}(\langle a|b\rangle\langle c|d\rangle+\langle a|c\rangle\langle b|d\rangle+\langle a|d\rangle\langle b|c\rangle)\big)\mathbf{1}.$$

The zero mode of the vacuum vector, $o(\mathbf{1})$, is the identity. So the weight-zero terms have trace on any V_k equal to the constant in the above expression times the dimension of V_k. Rewriting the constant to emphasize its symmetry, the weight-zero terms contribute

$$-\frac{n}{720}\big(4\langle[a,b]|[c,d]\rangle + 5\langle[a,d]|[b,c]\rangle\big) + \frac{n}{1152}\sum_{\pi\in S_4}\langle\pi(a)|\pi(c)\rangle\langle\pi(b)|\pi(d)\rangle \quad (24)$$

to the trace of $o(a[-1]b[-1]c[-1]d)$ on V_k, where $n = \dim V_k$. In particular when $n = \dim V_1$, this is the contribution of the weight-zero terms to $\operatorname{tr}\big|_{V_1} o(a[-1]b[-1]c[-1]d)$.

From the creation axiom, if v is of weight $k > 0$ then $o(v)\mathbf{1} = v(k-1)\mathbf{1} = 0$. As $V_0 = \mathbb{C}\mathbf{1}$, only the weight-zero terms contribute to the trace of $o(a[-1]b[-1]c[-1]d)$ on V_0. Since $\dim V_0 = 1$, the first part of the lemma, equation (20), follows immediately from equation (24).

For a weight-one vector, u, $o(u) = u(0) = \operatorname{ad} u$, the adjoint operator. Because V_1 is reductive [DM04b], $\operatorname{tr}\big|_{V_1} \operatorname{ad} u = 0$ for all $u \in V_1$. Thus the weight-one terms have no contribution to the trace on V_1.

We turn to the weight-two terms:

$$-\frac{1}{12}\Big(a(1)b(-1)c(-1)d + a(-1)b(1)c(-1)d + \langle c|d\rangle a(-1)b\Big) + \frac{1}{4}\Big(a(0)b(0)c(-1)d + a(0)b(-1)[c,d] + a(-1)\big[b,[c,d]\big]\Big). \quad (25)$$

Using the affine Lie algebra relations, we compute that

$$a(-1)b(1)c(-1)d = a(-1)[[b,c],d] + \langle b|c\rangle a(-1)d + \langle b|d\rangle a(-1)c,$$

$$\begin{aligned} a(1)b(-1)c(-1)d = {} & [[a,b],c](-1)d + c(-1)[[a,b],d] + \langle a|b\rangle c(-1)d \\ & + b(-1)[[a,c],d] + \langle a|c\rangle b(-1)d + \langle a|d\rangle b(-1)c. \end{aligned}$$

Each of the terms in these expansions has the form $u(-1)v$ for some u and v in V_1, as do the terms $\langle c|d\rangle a(-1)b$ and $a(-1)\big[b,[c,d]\big]$ from expression (25). So their zero modes have the form $\big(u(-1)v\big)(1)$. We use associativity, equation (1), to compute that

$$\big(u(-1)v\big)(1) = u(-1)v(1) + v(0)u(0) + v(-1)u(1)$$

as an operator on V_1. Thus

$$\operatorname{tr}\Big|_{V_1} \big(u(-1)v\big)(1) = \kappa(u,v) + 2\langle u|v\rangle.$$

For ease of notation, let $(u:v) = \kappa(u,v) + 2\langle u|v\rangle$. Note that $(u:v)$ is an invariant symmetric bilinear form, because both $\kappa(u,v)$ and $\langle u|v\rangle$ are.

There are two terms remaining in expression (25). One of these, $a(0)b(-1)[c,d]$ has the form $u(0)v(-1)w$, for some $u,v,w \in V_1$. We expand the other:

$$a(0)b(0)c(-1)d = a(0)[b,c](-1)d + a(0)c(-1)[b,d],$$

and see that its terms have the same form. The zero mode of this form is $\big(u(0)v(-1)w\big)(1)$.

$$\begin{aligned} \operatorname{tr}\Big|_{V_1} \big(u(0)v(-1)w\big)(1) &= \operatorname{tr}\Big|_{V_1} \big([u,v](-1)w\big)(1) + \operatorname{tr}\Big|_{V_1} \big(v(-1)[u,w]\big)(1) \\ &= ([u,v]:w) + (v:[u,w]) = 0 \end{aligned}$$

So $a(0)b(-1)[c,d]$ and $a(0)b(0)c(-1)d$ both have trace zero on V_1.

We now sum the traces of all the terms from expression (25) and get:

$$\begin{aligned} -\frac{1}{12}\Big(& -([a,d]:[b,c]) - ([b,d]:[a,c]) + ([a,b]:[c,d]) - ([c,d]:[a,b]) + \langle c|d\rangle(a:b) \\ & + \langle b|c\rangle(a:d) + \langle b|d\rangle(a:c) + \langle a|c\rangle(b:d) + \langle a|d\rangle(b:c)\Big) + \frac{1}{4}([a,b]:[c,d]). \end{aligned} \tag{26}$$

We wish to simplify this. Using the Lie-algebra Jacobi identity, it is easy to show that

$$([a,c]\!:\![b,d]) = ([a,b]\!:\![c,d]) + ([a,d]\!:\![b,c]), \tag{27}$$

for any invariant symmetric bilinear form. So expression (26) equals

$$\frac{1}{3}([a,b]\!:\![c,d]) + \frac{1}{6}([a,d]\!:\![b,c]) - \frac{1}{12}\Big(\langle a|b\rangle(c\!:\!d) + \langle a|c\rangle(b\!:\!d) \\ + \langle a|d\rangle(b\!:\!c) + \langle b|c\rangle(a\!:\!d) + \langle b|d\rangle(a\!:\!c) + \langle c|d\rangle(a\!:\!b)\Big).$$

Substituting $\kappa(u,v) + 2\langle u|v\rangle$ back in for $(u:v)$ and rewriting to emphasize the symmetry gives

$$\frac{2}{3}\langle[a,b]|[c,d]\rangle + \frac{1}{3}\langle[a,d]|[b,c]\rangle + \frac{1}{3}\kappa([a,b],[c,d]) + \frac{1}{6}\kappa([a,d],[b,c]) \\ - \frac{1}{24}\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\langle\pi(c)|\pi(d)\rangle - \frac{1}{48}\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\kappa(\pi(c),\pi(d)). \tag{28}$$

This is the contribution of the weight-two terms to $\mathrm{tr}\big|_{V_1} o(a[-1]b[-1]c[-1]d)$.

Now let's do the weight-three terms. These are:

$$\frac{1}{2}\Big(a(0)b(-1)c(-1)d + a(-1)b(0)c(-1)d + a(-1)b(-1)[c,d]\Big).$$

Expanding the first two terms this becomes

$$\frac{1}{2}\Big([a,b](-1)c(-1)d + b(-1)[a,c](-1)d + b(-1)c(-1)[a,d] \\ + a(-1)[b,c](-1)d + a(-1)c(-1)[b,d] + a(-1)b(-1)[c,d]\Big). \tag{29}$$

All the terms in the expansion have the form $u(-1)v(-1)w$ for some $u,v,w\in V_1$. The zero mode of this form is $\big(u(-1)v(-1)w\big)(2)$. We expand $(u(-1)v(-1)w)(2)$ as an operator on V_1 by applying associativity twice.

$$(u(-1)v(-1)w)(2) = \\ u(-1)w(1)v(0) + v(-1)w(1)u(0) + w(0)v(0)u(0) + w(-1)v(1)u(0).$$

Let x be in V_1 and compute that $w(-1)v(1)u(0)x = \langle v|[u,x]\rangle w$. Therefore

$$\mathrm{tr}\Big|_{V_1} w(-1)v(1)u(0) = \langle v|[u,w]\rangle.$$

Putting this together with the expansion of $(u(-1)v(-1)w)(2)$, we get

$$\operatorname{tr}\Big|_{V_1}(u(-1)v(-1)w)(2) = -\langle [u,v]|w\rangle + \operatorname{tr}\Big|_{V_1} w(0)v(0)u(0)). \tag{30}$$

Using equation (30) on each term in expression (29), we compute the contribution of the weight-three terms to $\operatorname{tr}\big|_{V_1} o(a[-1]b[-1]c[-1]d)$,

$$\frac{1}{2}\Bigg(-\langle [[a,b],c]|d\rangle - \langle [b,[a,c]]|d\rangle - \langle [b,c]|[a,d]\rangle - \langle [a,[b,c]]|d\rangle - \langle [a,c]|[b,d]\rangle$$
$$-\langle [a,b]|[c,d]\rangle + \operatorname{tr}\Big|_{V_1}\Big(d(0)c(0)[a,b](0) + d(0)[a,c](0)b(0) + [a,d](0)c(0)b(0)$$
$$+ d(0)[b,c](0)a(0) + [b,d](0)c(0)a(0) + [c,d](0)b(0)a(0)\Big)\Bigg).$$

To simplify this expression we use the properties of $\langle \ | \ \rangle$ including equation (27) on the first six terms; for the remaining terms, we expand the commutators using $[u,v](0) = u(0)v(0) - v(0)u(0)$ and then use $\operatorname{tr} AB = \operatorname{tr} BA$ to move $a(0)$ to the left of each term. We find the trace of the weight-three terms on V_1 is

$$-\langle [a,b]|[c,d]\rangle + \frac{1}{2}\operatorname{tr}\Big|_{V_1}\big(a(0)b(0)d(0)c(0) + a(0)c(0)d(0)b(0) - 2a(0)d(0)c(0)b(0)\big). \tag{31}$$

There is only one weight-four term, $a(-1)b(-1)c(-1)d$. We expand its zero mode as an operator on V_1 using associativity three times.

$$\big(a(-1)b(-1)c(-1)d)\big)(3) = a(-1)d(1)c(0)b(0) + b(-1)d(1)c(0)a(0)$$
$$+ c(-1)d(1)b(0)a(0) + d(0)c(0)b(0)a(0) + d(-1)c(1)b(0)a(0).$$

Now let x be in V_1; $a(-1)b(1)c(0)d(0)x = \langle b|[c,[d,x]]\rangle a$. Thus

$$\operatorname{tr}\Big|_{V_1} a(-1)b(1)c(0)d(0) = \langle [b,c]|[d,a]\rangle.$$

Using this, the expansion of $\big(a(-1)b(-1)c(-1)d\big)(3)$ and equation (27), we compute

$$\operatorname{tr}\Big|_{V_1}(a(-1)b(-1)c(-1)d)(3) = -\langle [a,b]|[c,d]\rangle - 2\langle [a,d]|[b,c]\rangle$$
$$+ \operatorname{tr}\Big|_{V_1} a(0)d(0)c(0)b(0). \tag{32}$$

This is the contribution of the weight-four term to $\operatorname{tr}\big|_{V_1} o(a[-1]b[-1]c[-1]d)$.

We now add the contributions of the round-bracket weights zero, two, three and four terms from expressions (24), (28), (31) and (32) to find that

$$\begin{aligned} \operatorname{tr}\Big|_{V_1} o(a[-1]b[-1]c[-1]d) &= -\frac{n+240}{720}\Big(4\langle [a,b]|[c,d]\rangle - 5\langle [a,d]|[b,c]\rangle\Big) \\ &+\frac{1}{3}\kappa([a,b],[c,d]) + \frac{1}{6}\kappa([a,d],[b,c]) + \frac{1}{2}\operatorname{tr}\Big|_{V_1}\big(a(0)b(0)d(0)c(0) \\ &+a(0)c(0)d(0)b(0)\big) + \frac{n-48}{1152}\textstyle\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\langle \pi(c)|\pi(d)\rangle \\ &-\frac{1}{48}\textstyle\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\kappa(\pi(c),\pi(d)) \end{aligned} \tag{33}$$

(Recall that the trace of the weight-one terms on V_1 is zero.) We uncover another symmetry in this trace by expanding $\kappa([a,b],[c,d])$ and $\kappa([a,d],[b,c])$.

$$\begin{aligned} \kappa([a,b],[c,d]) &= \operatorname{tr}\Big|_{V_1}[a,b](0)[c,d](0) \\ &= \operatorname{tr}\Big|_{V_1}\big(a(0)b(0)c(0)d(0) - a(0)b(0)d(0)c(0) \\ &\quad - a(0)c(0)d(0)b(0) + a(0)d(0)c(0)b(0)\big) \end{aligned}$$

Likewise,

$$\begin{aligned} \kappa([a,d],[b,c]) &= \operatorname{tr}\Big|_{V_1}\big(a(0)d(0)b(0)c(0) - a(0)d(0)c(0)b(0) \\ &\quad - a(0)b(0)c(0)d(0) + a(0)c(0)b(0)d(0)\big). \end{aligned}$$

So,

$$\begin{aligned} \frac{1}{3}\kappa([a,b],[c,d])+\frac{1}{6}\kappa([a,d],[b,c])+\frac{1}{2}\operatorname{tr}\Big|_{V_1}\big(a(0)b(0)d(0)c(0)+a(0)c(0)d(0)b(0)\big) \\ = \frac{1}{6}\sum_{\pi\in S_3} a(0)\pi\big(b(0)\big)\pi\big(c(0)\big)\pi\big(d(0)\big). \end{aligned}$$

Substituting this into equation (33) completes the proof of the second equation in the lemma. □

In order to define elements of $V_{[4]}$ with zero trace on V_0 and non-zero trace on V_1, we need to know about the structure of V_1. We will use results from Dong and Mason's paper *Holomorphic vertex operator algebras of small central charge*, which we summarize here. [DM04a]

Dong and Mason studied the structure of strongly-rational holomorphic VOAs with central charges, c, equal to 8, 16 or 24. As we have already completed the proof of Theorem 1.1 in the $c = 8$ case, we need to know about $c = 16$ and $c = 24$. For $c = 16$, Dong and Mason show that we have a lattice VOA, V_L, where L is one of the two unimodular rank 16 lattices $E_8 + E_8$ or D_{16}^+. The $E_8 + E_8$ lattice VOA has V_1 isomorphic to the Lie algebra of type $E_8 \oplus E_8$ and the D_{16}^+ lattice VOA has V_1 isomorphic to the Lie algebra of type D_{16}. For $c = 24$, Dong and Mason find that V_1 is either 0, abelian of rank 24 or semi-simple of rank less than or equal to 24.

We will deal with the two $c = 16$ cases and all of the $c = 24$ cases except $V_1 = 0$, V_1 abelian or V_1 simple of type A_1, A_2, D_4, E_6, E_7, E_8, G_2, or F_4 using Lemmas 6.4 and 6.5. Because we will frequently refer to the list of exceptional Lie algebras plus A_1, A_2 and D_4, we will call this list the *augmented exceptional list* of Lie algebras. We address V_1 on the augmented exceptional list using Lemma 6.8 and V_1 abelian using Lemma 6.9. We do not consider the $V_1 = 0$ case, as our method uses elements of V_1. Dong and Mason describe the space of graded traces for the Moonshine module, for which $V_1 = 0$. [DM00] It is conjectured that this is the only $c = 24$, holomorphic VOA with $V_1 = 0$. [FLM88].

Remark 6.2. *This augmented exceptional list of Lie algebras is the same list considered by Deligne in his paper "La Série exceptionnelle de groupes de Lie." [Del96]*

The elements of $V_{[4]}$ used in Lemmas 6.4 and 6.5 are defined to have zero trace on V_0. We start by choosing u and v in V_1, orthogonal and of the same length with respect to the Li-Zamilodichov metric and commuting with respect to the Lie bracket. We then define $x(u, v) \in V_{[4]}$:

$$x(u, v) := u[-1]^3 u - 6u[-1]^2 v[-1]v + v[-1]^3 v. \tag{34}$$

It is now a straight-forward application of Lemma 6.1 to compute the traces of $x(u, v)$ on V_0 and V_1.

Lemma 6.3. *Let V be a strongly rational VOA, let u and v be orthogonal commuting elements of V_1 such that $\langle u|u\rangle = \langle v|v\rangle$, and let $x(u, v)$ be as defined in equation* (34) *above. Then*

$$\begin{aligned}
&\operatorname{tr}\big|_{V_0} o(x(u, v)) = 0, \\
&\operatorname{tr}\big|_{V_1} o(x(u, v)) = \operatorname{tr}\big|_{V_1} \left(u(0)^4 - 6u(0)^2 v(0)^2 + v(0)^4\right).
\end{aligned}$$

In the case that V_1 is semisimple but not on the augmented exceptional list, we show that there exist u and v satisfying the conditions of Lemma 6.3 such that the trace of $o(x(u,v))$ on V_1 is nonzero. We choose linearly independent u and v in a Cartan subalgebra, $\mathfrak{h}$, of V_1, which is possible unless V_1 is simple of type A_1. Such u and v commute and it is easy to compute $\operatorname{tr}|_{V_1}(u(0)^4 - 6u(0)^2v(0)^2 + v(0)^4)$. Indeed, suppose $\mathfrak{h}$ has root system Φ and root spaces L_α. The Cartan decomposition of $\mathfrak{g}$ is: $V_1 = \mathfrak{h} \oplus \sum_{\alpha\in\Phi} L_\alpha$. Clearly, for any $u \in \mathfrak{h}$, $u(0)x = [u,x] = 0$ for any x in $\mathfrak{h}$, while $u(0)x = [u,x] = \alpha(u)x$ for any x in L_α. As each L_α is one dimensional, it follows that: For any u and v in the Cartan subalgebra of V_1,

$$\operatorname{tr}\big|_{V_1}(u(0)^4 - 6u(0)^2v(0)^2 + v(0)^4) = \sum_{\alpha\in\Phi} \alpha(u)^4 - 6\alpha(u)^2\alpha(v)^2 + \alpha(v)^4 \quad (35)$$

In addition to commuting, we need u and v to be orthogonal and of the same length. The semisimple Lie algebra V_1 has a decomposition into simple Lie algebras.

$$V_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad (36)$$

Such a decomposition is orthogonal with respect to any invariant bilinear form, in particular the Li-Zamilodichov metric.

In our first case, V_1 semi-simple but not simple, i.e., $m > 1$. We guarantee the orthogonality of u and v by choosing $u \in \mathfrak{g}_1$ and $v \in \mathfrak{g}_2$. We will actually pick u and v in real subspaces of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, the Cartan subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Let Φ_i be the root system of $\mathfrak{g}_i$. For each root $\alpha \in \Phi_i$, fix a non-zero $e_\alpha \in L_\alpha$. There exist $f_\alpha \in L_{-\alpha}$ and $h_\alpha := [e_\alpha, f_\alpha] \in \mathfrak{h}_i$ such that $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, f_\alpha] = -2f_\alpha$. Define $\mathfrak{r}_i = \operatorname{Span}_{\mathbb{R}}\{h_\alpha | \alpha \in \Phi_i\}$. The Killing form is a real-valued positive-definite form on $\mathfrak{r}_i$. Furthermore, Dong and Mason prove that, for strongly rational, holomorphic VOAs with central charge at most 24,

$$\kappa(u,v) = 2\langle u|v\rangle\big(\frac{n}{c} - 1\big), \quad (37)$$

where $n = \dim V_1$ [DM04a]. In particular, as c is an integer and for V_1 semisimple both forms are nondegenerate on V_1, $\langle\ |\ \rangle$ is a non-zero rational multiple of the Killing form. Hence it is also positive definite on each $\mathfrak{r}_i$. Thus we can choose u in $\mathfrak{r}_1$ and v in $\mathfrak{r}_2$ such that $\langle u|u\rangle = \langle v|v\rangle \neq 0$.

Lemma 6.4. *Let V be a strongly rational holomorphic VOA with $c \leq 24$, and suppose that V_1 is semisimple, but not simple. Then there exist u and v*

in V_1 such that

$$\mathrm{tr}\big|_{V_0} o(x(u,v)) = 0,$$
$$\mathrm{tr}\big|_{V_1} o(x(u,v)) \neq 0.$$

Proof. From the above discussion there exist $u \in \mathfrak{r}_1$ and $v \in \mathfrak{r}_2$ such that $\langle u|u\rangle = \langle v|v\rangle \neq 0$. As $\mathfrak{r}_i \subset \mathfrak{g}_i$, u and v are commuting and orthogonal. Thus u and v satisfy the conditions of Lemma 6.3. It follows immediately that $\mathrm{tr}\big|_{V_0} o(x(u,v)) = 0$, and, since $\mathfrak{r}_i \subset \mathfrak{h}_i$, equation (35) implies that

$$\mathrm{tr}\big|_{V_1} o(x(u,v)) = \sum_{\alpha\in\Phi} \alpha(u)^4 - 6\alpha(u)^2\alpha(v)^2 + \alpha(v)^4,$$

where Φ is a root system for V_1. Let Φ_i be the root system for $\mathfrak{g}_i$. Then $\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_m$, and $\alpha(u) = 0$, for any $\alpha \notin \Phi_1$, likewise, $\alpha(v) = 0$ for any $\alpha \notin \Phi_2$. Thus

$$\mathrm{tr}\big|_{V_1} o(x(u,v)) = \sum_{\alpha\in\Phi_1} \alpha(u)^4 + \sum_{\alpha\in\Phi_2} \alpha(v)^4.$$

For all roots α and β, $\alpha(h_\beta)$ is an integer. Since u and v are in the real span of the h_β's, $\alpha(u)$ and $\alpha(v)$ are both real numbers. Hence, $\mathrm{tr}\big|_{V_1} o(x(u,v)) \geq 0$. Furthermore, because $u \neq 0$ and Φ_1 spans $\mathfrak{h}_1^*$, there exists an $\alpha \in \Phi_1$ such that $\alpha(u) \neq 0$. Hence $\mathrm{tr}\big|_{V_1} o(x(u,v)) > 0$. □

We are left with the cases V_1 abelian and V_1 simple. If V_1 is abelian, then we cannot use Lemma 6.3 to get $\mathrm{tr}\big|_{V_1} o(x(u,v)) \neq 0$, but we can use this lemma to address all the V_1 simple cases except those on the augmented exceptional list.

Assume V_1 is simple, but not on the augmented exceptional list. We choose u and v in the Cartan subalgebra of V_1 so that $[u,v] = 0$. We also need u and v orthogonal and of equal length with respect to the form $\langle\ |\ \rangle$. As we mentioned previously, for V strongly rational and holomorphic with $c \leq 24$ and V_1 semisimple, $\langle\ |\ \rangle$ is a non-zero rational multiple of the Killing form on V_1. As V_1 is not A_1 type, the dimension of the Cartan subalgebra is greater than one, and we can choose u and v orthogonal and of the same length with respect to the Killing form. They will have the same properties with respect to $\langle\ |\ \rangle$.

Lemma 6.5. *Let V be a strongly rational VOA and suppose that V_1 is a simple Lie algebra of type A, B, C or D but not type A_1, A_2 or D_4. There exist u and v in V_1 such that*

$$\operatorname{tr}\big|_{V_0} o(x(u,v)) = 0$$
$$\operatorname{tr}\big|_{V_1} o(x(u,v)) \neq 0$$

Proof. For each root type we pick explicit u and v in the Cartan subalgebra $\mathfrak{h}$ of V_1, orthogonal and of equal length with respect to the Killing form. From our discussion above, such a u and v satisfy the conditions of Lemma 6.3, thus the trace of $o(x(u,v))$ on V_0 is 0.

Let $(\ ,\ )$ be the nondegenerate bilinear form induced on $\mathfrak{h}^*$ by the Killing form. Then for u in $\mathfrak{h}$ and α in $\mathfrak{h}^*$, $\alpha(u) = (\alpha, u^*)$. Using this, Lemma 6.3 and equation (35), we have:

$$\operatorname{tr}\big|_{V_1} o(x,(u,v)) = \sum_{\alpha\in\Phi} (\alpha,u^*)^4 - 6(\alpha,u^*)^2(\alpha,v^*)^2 + (\alpha,v^*)^4, \qquad (38)$$

where Φ is the root system for V_1. Letting $\{e_1, e_2, \ldots, e_\ell\}$ be an orthonormal set, the root systems of type A, B, C and D can be realized as follows.

		Root system
A_ℓ	$\ell \geq 1$	$\{\pm(e_i - e_j) \mid 1 \leq i < j \leq \ell+1\}$
B_ℓ	$\ell \geq 2$	$\{\pm e_i, \pm(e_i \pm e_j) \mid 1 \leq i < j \leq \ell\}$
C_ℓ	$\ell \geq 3$	$\{\pm 2e_i, \pm(e_i \pm e_j) \mid 1 \leq i < j \leq \ell\}$
D_ℓ	$\ell \geq 4$	$\{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq \ell\}$

For A_ℓ, $\ell \geq 3$, choose u so that $u^* = (e_1 - e_2)$ and v so that $v^* = (e_\ell - e_{\ell+1})$. Since $\ell \geq 3$, u and v are perpendicular. Computing using equation (38),

$$\operatorname{tr}\big|_{A_\ell(\mathbb{C})} o(x(u,v)) = 8\ell + 8.$$

So for $\ell \geq 3$, $\operatorname{tr}\big|_{A_\ell(\mathbb{C})} o(x(u,v)) \neq 0$. For B_ℓ, C_ℓ and D_ℓ, we choose u so that $u^* = e_1$ and v so that $v^* = e_\ell$. From equation (38),

$$\operatorname{tr}\big|_{B_\ell(\mathbb{C})} o(x(u,v)) = 8\ell - 28,$$
$$\operatorname{tr}\big|_{C_\ell(\mathbb{C})} o(x(u,v)) = 8\ell + 32,$$
$$\operatorname{tr}\big|_{D_\ell(\mathbb{C})} o(x(u,v)) = 8\ell - 32.$$

Therefore $\operatorname{tr}\big|_{B_\ell(\mathbb{C})}\upsilon(u,v) \neq 0$ for any integer ℓ, $\operatorname{tr}\big|_{C_\ell(\mathbb{C})}\upsilon(u,v) \neq 0$ for any positive ℓ and $\operatorname{tr}\big|_{D_\ell(\mathbb{C})}\upsilon(u,v) \neq 0$ for $\ell \neq 4$. □

Remark 6.6. *One can actually show that for the V_1 on the augmented exceptional list, it is not possible to choose u and v as in Lemma 6.5 so that the trace of $x(u,v)$ on V_1 is non-zero. For a proof see [Hur02]. In the type A, D and E cases, this reflects the fact that A_1, A_2, D_4, E_6, E_7 and E_8 are the strongly perfect root lattices, i.e., those root lattices which are spherical 4-designs. [Mar03]*

Both $c = 16$ cases; V_1 of type $E_8 + E_8$ and V_1 of type D_{16} have been addressed, by Lemmas 6.4 and 6.5 respectively. Thus we are left with the V_1 on the augmented exceptional list with $c = 24$

Recently, in their paper *Integrability of C_2-cofinite vertex operator algebras*, Dong and Mason prove that the levels of the affine Lie algebras created by the simple subalgebras of V_1 must be positive integers. [DM] We do not define level here, but once one understands it, it is a relatively simple computation to show that for $c = 24$ VOAs with V_1 on the augmented exceptional list, only D_4-type V_1 has an integral level. To do this one uses the values in table 6. Thus most of the cases we will address in Lemma 6.8 are actually moot. In fact, as a VOA with V_1 of D_4 type does not appear on Schellekens list of holomorphic $c = 24$ VOAs [Sch93], the D_4 case is likely moot as well. However, as we do not want to rely on Schellekens list, which is not mathematically rigorous, we proceed with the D_4 case. And, since it merely adds some lines to table 6, which we want to include to aid the reader in his or her computation of the levels, we proceed with the other augmented-exceptional cases as well.

To address the $c = 24$ VOAs with V_1 on the augmented exceptional list, we need a different approach to that taken in Lemmas 6.4 and 6.5. Recall equation (37), relating the Killing form to the Li-Zamilodichov metric, for strongly rational holomorphic VOAs with central charge less than 24 and V_1 semisimple. For $c = 24$, it yields

$$\langle u|v\rangle = \frac{12}{n-24}\kappa(u,v), \tag{39}$$

where n is the dimension of V_1. (Note that this implies that $n \neq 24$ for V_1 semi-simple. In fact, $n = 24$ if and only if V_1 is abelian [DM04a].) Using equation (39) to rewrite Lemma 6.3 in terms of the Killing form yields the following lemma.

Lemma 6.7. *Let V be a strongly rational holomorphic $c = 24$ VOA with V_1 semisimple, and let a, b, c and d be in V_1. Then*

$$\begin{aligned}
\operatorname{tr}\big|_{V_0} o(a[-1]b[-1]c[-1]d) &= -\frac{1}{60(n-24)}\big(4\kappa([a,b],[c,d]) + 5\kappa([a,d],[b,c])\big) \\
&\quad + \frac{1}{8(n-24)^2}\sum_{\pi\in S_4}\kappa(\pi(a),\pi(b))\kappa(\pi(c),\pi(d)), \\
\operatorname{tr}\big|_{V_1} o(a[-1]b[-1]c[-1]d) &= -\frac{n+240}{60(n-24)}\big(4\kappa([a,b],[c,d]) + 5\kappa([a,d],[b,c])\big) \\
&\quad - \frac{n}{8(n-24)^2}\sum_{\pi\in S_4}\kappa(\pi(a),\pi(b))\kappa(\pi(c),\pi(d)) \\
&\quad + \frac{1}{6}\sum_{\pi\in S_3}\operatorname{tr}\Big|_{V_1} a(0)\pi\big(b(0)\big)\pi\big(c(0)\big)\pi\big(d(0)\big).
\end{aligned}$$

For each root α of V_1 there exist $e \in L_\alpha$, $f \in L_{-\alpha}$ and $h \in \mathfrak{h}$ satisfying the standard $\mathfrak{sl}_2$ commutation relations:

$$[e,f] = h \qquad [h,e] = 2e \qquad [h,f] = -2f.$$

From Lemma 6.7, we compute

$$\operatorname{tr}\Big|_{V_0} o(h[-1]^3h) = \frac{3\kappa(h,h)^2}{(n-24)^2}, \tag{40}$$

$$\operatorname{tr}\Big|_{V_1} o(h[-1]^3h) = -\frac{3n\kappa(h,h)^2}{(n-24)^2} + \operatorname{tr}\big|_{V_1} h(0)^4, \tag{41}$$

and

$$\operatorname{tr}\Big|_{V_0} o(e[-1]f[-1]e[-1]f) = \frac{\kappa(h,h)}{60(n-24)} + \frac{\kappa(h,h)^2}{2(n-24)^2}, \tag{42}$$

$$\operatorname{tr}\Big|_{V_1} o(e[-1]f[-1]e[-1]f) = \frac{\kappa(h,h)(n+240)}{60(n-24)} - \frac{n\kappa(h,h)^2}{2(n-24)^2} + \frac{1}{6}\operatorname{tr}\big|_{V_1} h(0)^4. \tag{43}$$

For the traces of $o(e[-1]f[-1]e[-1]f)$ we use $2\kappa(e,f) = \kappa(h,h)$, which follows easily from the invariance of the Killing form, the commutation relations and

$$\operatorname{tr}\big|_{V_1}\big(2e(0)f(0)e(0)f(0) + 4e(0)^2f(0)^2\big) = \operatorname{tr}\big|_{V_1} h(0)^4. \tag{44}$$

To prove equation (44), we decompose V_1 as a module for $\mathfrak{s}_\alpha$, the subalgebra generated by e, f and h, $V_1 = M_1 \oplus M_2 \oplus \cdots \oplus M_m$. Each M_i is an irreducible $\mathfrak{sl}_2$-module with dimension at most 4. For each dimension, such a module is unique up to isomorphism and completely known. So we can just check that $\operatorname{tr}\big|_{M_i}\big(2e(0)f(0)e(0)f(0) + 4e(0)^2 f(0)^2\big) = \operatorname{tr}\big|_{M_i} h(0)^4$ holds for each of them.

Now define C to be $\frac{\operatorname{tr}|_{V_0} o(e[-1]f[-1]e[-1]f)}{\operatorname{tr}|_{V_0} o(h[-1]^3 h)}$. From equations (42) and (40),

$$C = \frac{n-24}{180\kappa(h,h)} + \frac{1}{6}.$$

Define $y(\alpha)$ to be

$$y(\alpha) = e[-1]f[-1]e[-1]f - Ch[-1]^3 h. \tag{45}$$

The constant C is defined so that

$$\operatorname{tr}\big|_{V_0} o(y(\alpha)) = 0.$$

It is straight forward to compute the trace of $o(y(\alpha))$ on V_1 using equations (41) and (43);

$$\operatorname{tr}\big|_{V_1} o(y(\alpha)) = \frac{\kappa(h,h)(n+120)}{30(n-24)} - \frac{n-24}{180\kappa(h,h)} \operatorname{tr}\big|_{V_1} h(0)^4. \tag{46}$$

We wish to show that, for $c = 24$ VOAs with V_1 on the augmented exceptional list, there exists a root α such that $\operatorname{tr}\big|_{V_1} o(y(\alpha)) \neq 0$. To simplify the computation, we choose α to be a long root, in which case

$$\operatorname{tr}\big|_{V_1} h(0)^4 = \kappa(h,h) + 24. \tag{47}$$

To show this we again decompose V_1 into irreducible modules for $\mathfrak{s}_\alpha$

$$V_1 = \mathfrak{s}_\alpha \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_k.$$

Recall that $\kappa(h,h) = \operatorname{tr}\big|_{V_1} h(0)^2$. This trace, like the trace of $h(0)^4$, splits over the decomposition. Since α is a long root, $\dim M_i \leq 2$ for all i, and we compute that

$$\operatorname{tr}\big|_{M_i} h(0)^2 = \operatorname{tr}\big|_{M_i} h(0)^4,$$
$$\operatorname{tr}\big|_{\mathfrak{s}_\alpha} h(0)^2 + 24 = \operatorname{tr}\big|_{\mathfrak{s}_\alpha} h(0)^4.$$

Equation (47) follows.

Inserting equation (47) into equation (46) and factoring, we see that, for a long root α, $\operatorname{tr}\big|_{V_1} o(y(\alpha)) = 0$ if and only if

$$\big(6\kappa(h,h) - n + 24\big)\big((n+120)\kappa(h,h) + 24(n-24)\big) = 0.$$

Since n must be positive and $\kappa(h,h) \geq 8$, the second factor cannot be zero. Thus the trace of $o(y(\alpha))$ on V_1 is zero if and only if the first factor is zero. Table 6 gives the values of n and $\kappa(h,h)$ for the on the augmented list of exceptional Lie algebras. Clearly no pair of values satisfies $n = 6\kappa(h,h)+24$. This gives us the following lemma.

V_1	$\dim V_1$	$\kappa(h,h)$
A_1	3	8
A_2	8	12
D_4	28	24
E_6	78	48
E_7	133	72
E_8	248	120
F_4	52	36
G_2	14	16

Lemma 6.8. *Let V be a strongly rational homomorphic $c = 24$ vertex operator algebra with V_1 simple of type A_1, A_2, D_4 or exceptional. Then for a long root α of V_1,*

$$\operatorname{tr}\big|_{V_0} o(y(\alpha)) = 0,$$
$$\operatorname{tr}\big|_{V_1} o(y(\alpha)) \neq 0.$$

See equation (45) *for the definition of $y(\alpha)$.*

We turn to the V_1 abelian case. Dong and Mason show that in this case V is the lattice VOA generated by the Leech lattice. [DM04a] As the Killing form on an abelian Lie algebra is zero, while the Li-Zamilodichov metric is non-degenerate on V_1, equation (37) implies that, for strongly rational

holomorphic $c = 24$ VOAs with V_1 abelian, $\dim V_1 = n = 24$. Let a be an element of V_1, and rewrite $a[-1]^3 a$ as $a[-1]^4\mathbf{1}$. From Lemma 6.1

$$\begin{aligned} \operatorname{tr}\big|_{V_0} o(a[-1]^4\mathbf{1}) &= \frac{1}{48}\langle a|a\rangle^2, \\ \operatorname{tr}\big|_{V_1} o(a[-1]^4\mathbf{1}) &= -\frac{1}{2}\langle a|a\rangle^2. \end{aligned} \tag{48}$$

Next we compute the traces of $a[-2]^2\mathbf{1}$. For this we use the following equation of Zhu:

$$Z(a[-2]b,\tau) = -\sum_{k=2}^{\infty}(2k-1)E_{2k}(\tau)Z(a[2k-2]b,\tau)$$

[Zhu96, Prop. 4.3.6]; see also [DMN01, Prop 2.2.1]. Let $b = a[-2]\mathbf{1}$. As $a[2k-2]a[-2]\mathbf{1} = 0$ for $k \geq 3$, the above equation becomes

$$Z(a[-2]^2\mathbf{1},\tau) = -3E_4(\tau)Z(a[2]a[-2]\mathbf{1},\tau).$$

Furthermore, from the affine Lie algebra commutation relations $a[2]a[-2]\mathbf{1} = 2\langle a|a\rangle\mathbf{1}$, so

$$\begin{aligned} Z(a[-2]^2\mathbf{1},\tau) &= -6\langle a|a\rangle E_4(\tau)Z(\mathbf{1},\tau), \\ &= -6\langle a|a\rangle(\frac{1}{720} + \frac{1}{3}q + \cdots)(q^{-1} + 24 + \cdots), \\ &= -\frac{\langle a|a\rangle}{120}q^{-1} - \frac{11\langle a|a\rangle}{5} + \cdots . \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{tr}\big|_{V_0} o(a[-2]^2\mathbf{1}) &= -\frac{\langle a|a\rangle}{120}, \\ \operatorname{tr}\big|_{V_1} o(a[-2]^2\mathbf{1}) &= -\frac{11\langle a|a\rangle}{5}. \end{aligned} \tag{49}$$

Lemma 6.9. *Let V be a strongly holomorphic $c = 24$ VOA with V_1 abelian. Then there exists an element a in V_1 such that*

$$\begin{aligned} \operatorname{tr}\big|_{V_0} o((2a[-1]^4 + 5\langle a|a\rangle a[-2]^2)\mathbf{1}) &= 0, \\ \operatorname{tr}\big|_{V_1} o((2a[-1]^4 + 5\langle a|a\rangle a[-2]^2)\mathbf{1}) &\neq 0. \end{aligned}$$

Proof. Since the Li-Zamilodichov metric is nondegenerate on V_1, there exists an $a \in V_1$ such that $\langle a|a\rangle \neq 0$. From equations (48) and (49), we compute that

$$\begin{aligned} \operatorname{tr}\big|_{V_0} o((2a[-1]^4 + 5\langle a|a\rangle a[-2]^2)\mathbf{1}) &= 0, \\ \operatorname{tr}\big|_{V_1} o((2a[-1]^4 + 5\langle a|a\rangle a[-2]^2)\mathbf{1}) &= -12\langle a|a\rangle^2. \end{aligned}$$

□

This completes the last case in the proof of the main theorems.

7 Conclusion

Having seen that for strongly rational holomorphic VOAs of small central charge there are no nontrivial obstructions to attaining the expected modular forms as graded traces, we can explore the question of which modular forms can be attained as the graded traces of highest-weight vectors. Dong, Mason and Nagatomo conjecture that there are no nontrivial obstructions in that case either. [DMN01] They prove their conjecture in the $c = 8$ case and produce an interesting class of highest-weight vectors for lattice VOAs using spherical harmonics and compute their graded traces. [DMN01] For the moonshine module, Dong and Mason find another class of highest-weight vectors with weight divisible by four and compute their graded traces. [DM00]. For the same VOA, the author finds a related family of highest-weight vectors and their graded traces. [Hur03] Together these papers show that for the moonshine module all cusp forms with weights 12, 16, 18, 20, 22 and 26 can be attained as graded traces. (In the moonshine module cases, $V_1 = 0$ so the forms must be cusp forms.)

Dong, Mason and Nagatomo's use of spherical harmonics to produce highest-weight vectors for lattice VOAs can be generalized to produce highest-weight vectors for all VOAs of strong CFT-type. In fact, the vector $x(u, v)$ used in Lemma 6.3 is an example of such a highest-weight vector. [Hur02] As in this paper, one can compute their traces on V_0 and V_1 relative to the root system of V_1, but this only determines the graded traces when the weight is low enough for the relevant space of modular forms to be one dimensional. For example, in the $c = 24$, $V_1 \neq 0$ case, the methods of this paper should be able to produce highest-weight vectors with specific graded traces for weights 6, 8 and 10, but not for higher weights. To compute the graded traces of given

highest-weight vectors for general VOAs is difficult as one needs knowledge of the structure of V_n for all n. Completion of the classification of holomorphic VOAs of small central charge would make this project feasible in those cases. [Sch93], [DM04a], [DM].

8 Acknowledgments

The author would like to thank her dissertation adviser, Professor Geoffrey Mason, and the other members of her dissertation committee, Professors Chongying Dong and Michael Tuite.

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